sufficient to keep the flamesheet from collapsing onto the surface. Inadequately fueled flows can occur in both the weak and strong interaction limits and can be described in terms closely analogous to those for a partly catalytic wall. Surface flamesheet flows are not presented here. In the strong interaction case, $C$ and $f_w$ were assumed, and $(f_w+\eta_*)2^{-1/2}$ was found as the inverse of the complementary error function, Eq. (30). The second essential parameter, $N^2/k\hat{\chi}g_C$, was then found from Eq. (31b). Equation (22b) gives $M\delta^*/x$. Also, $C_{H*}Re_\infty^{1/2}/(g_C N) = \phi_4$.

In both the weak and strong interaction limits, the pressure, displacement, energy of combustion, and flamesheet ordinate, which appear in the numerator of the ordinates plotted in Fig. 2, are roughly proportional to the product of the classical interaction parameter $\hat{\chi}$ and the energy parameter $g_C$. The product $\hat{\chi}g_C$ appears, rather than $\hat{\chi}$ alone, because the characteristic fluid enthalpy in the boundary layer is defined by the combustion process rather than the freestream stagnation enthalpy and surface conduction. Both $g_C$ and $g_w$ would have appeared in a linear form, e.g., Eq. (26), had our application allowed a cooled surface. Note also that injection of a light fuel (small $W_{fu}$ and $C$) has a larger effecton these properties than a heavy fuel.


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# Heat and Mass Transfer on Cones at Angles of Attack


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THIS Note summarizes a parametric study of laminar boundary-layer flows at the most windward generators of sharp cones at angles of attack. Several model gases and equilibrium air flows are considered with Mach numbers, wall to total enthalpy ratios and cross-flows parameters spanning the ranges of main engineering interest. Exact boundary-layer calculations are employed.

Starting with the pioneering work of Moore[1] boundary-layer flow on sharp cones at angles of attack in supersonic flow has received considerable attention as a fluid mechanics rather than a heat-transfer problem. Using exact, constant density-viscosity product solutions with $Pr = 0.7$ for low speed flows and $Pr = 1.0$ for general flows, Reshotko[2] suggested a $Pr^{0.37}$ and $Pr^{0.5}$ variation of heat-transfer and recovery factors, respectively. Some exact solutions,[3,4] based on experimentally determined pressure distributions and normalized by the leading cone generator values were found to be in good agreement with test data. Mass transfer effects on slender cones were studied by Fannelop and Smith,[5] under the assumption of small cross-flow.

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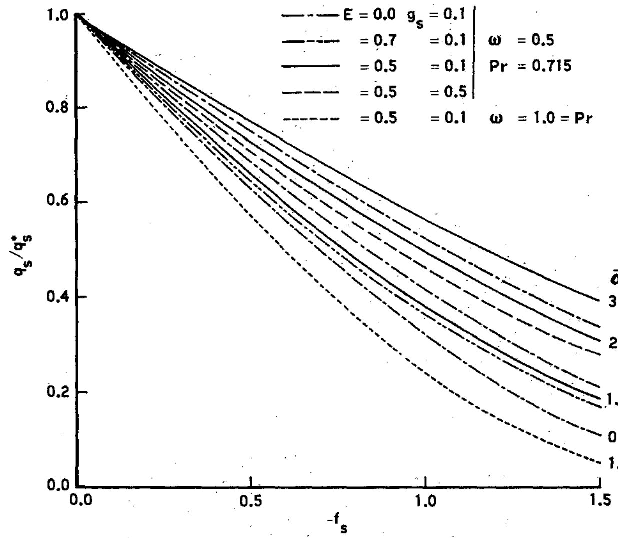


**Fig. 1 Reduction of heat transfer by injection.**

In a parametric study illustrating the salient features of the flow, Libby[6] employed the assumptions of a constant density-viscosity product and Prandtl number of unity, thus concerning himself primarily with the fluid mechanics aspects of the problem rather than heat transfer which is generally critically dependent on variations of gas properties. In contrast to Libby's work the main interest here is directed at the display of the influence of variation of gas properties.

**Table 1 Influence of gas properties on $q_s^*/q_{s,0}$**

| $\bar{\alpha}$ | $Pr$ / $\omega$ | 0.715 / 0.5 | 0.715 / 0.7 | 0.715 / 1.0 | 1.00 / 1.00 | 10,000 fps | 20,000 fps | 25,000 fps |
|---|---|---|---|---|---|---|---|---|
| | | a) $g_s = 0.10$, $E_1 = 0.5$ | | | | | | |
| 0.0 | | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.5 | | 1.30 | 1.29 | 1.29 | 1.30 | 1.30 | 1.31 | 1.30 |
| 1.0 | | 1.53 | 1.53 | 1.52 | 1.54 | 1.54 | 1.55 | 1.54 |
| 2.0 | | 1.92 | 1.91 | 1.90 | 1.94 | 1.94 | 1.94 | 1.94 |
| 3.0 | | 2.24 | 2.24 | 2.22 | 2.26 | 2.26 | 2.28 | 2.26 |
| | | b) $g_s = 0.10$, $E_2 = 0.7$ | | | | | | |
| 0.0 | | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | | 1.00 |
| 0.5 | | 1.33 | 1.33 | 1.33 | 1.34 | 1.33 | | 1.33 |
| 1.0 | | 1.60 | 1.59 | 1.59 | 1.61 | 1.59 | | 1.58 |
| 2.0 | | 2.02 | 2.01 | 2.00 | 2.04 | 2.00 | | 1.99 |
| 3.0 | | 2.36 | 2.35 | 2.34 | 2.39 | 2.35 | | 2.34 |
| | | c) $g_s = 0.10$, $E_1 = 0.9$ | | | | | | |
| 0.00 | | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | | |
| 0.5 | | 1.48 | 1.48 | 1.48 | 1.51 | 1.49 | | |
| 1.0 | | 1.78 | 1.81 | 1.82 | 1.85 | 1.80 | | |
| 2.0 | | 2.29 | 2.34 | 2.34 | 2.38 | 2.31 | | |
| 3.0 | | 2.70 | 2.76 | 2.76 | 2.81 | 2.75 | | |
| | | c) $g_s = 0.50$, $E_1 = 0.7$ | | | | | | |
| 0.0 | | 1.00 | 1.00 | 1.00 | 1.00 | | | |
| 0.5 | | 1.41 | 1.42 | 1.42 | 1.42 | | | |
| 1.0 | | 1.71 | 1.73 | 1.72 | 1.72 | | | |
| 2.0 | | 2.19 | 2.20 | 2.21 | 2.19 | | | |
| 3.0 | | 2.58 | 2.60 | 2.60 | 2.58 | | | |
| | | d) $g_s = 0.05$, $E_1 = 0.7$ | | | | | | |
| 0.0 | | 1.00 | 1.00 | 1.00 | 1.00 | | | |
| 0.5 | | 1.32 | 1.32 | 1.32 | 1.34 | | | |
| 1.0 | | 1.58 | 1.57 | 1.57 | 1.00 | | | |
| 2.0 | | 1.99 | 1.98 | 1.97 | 2.01 | | | |
| 3.0 | | 2.33 | 2.32 | 2.31 | 2.36 | | | |

The self-similar, transformed governing differential equations for boundary-layer flow at the leading cone generator are[7]

$$(Cf_1'')' + (f_1 + \bar{\alpha} f_2) f_1'' = 0 \quad (1a)$$

$$(Cf_2'')' + (f_1 + \bar{\alpha} f_2) f_2'' + \tfrac{2}{3}(\rho_e/\rho - f_1' f_2') + \bar{\alpha}(\rho_e/\rho - f_2'^2) = 0 \quad (1b)$$

$$[(C/Pr)g']' + (f_2 + \bar{\alpha} f_2)g' + E_1\{2C[(Pr-1)/Pr] f_1' f_1''\}' = 0 \quad (1c)$$

with the boundary conditions:

$$\eta = 0 \quad f_1' = 0 = f_2'; \quad f_1 = f_s; \quad f_2 = 0; \quad g = g_s \quad (2a)$$

$$\eta \to \infty \quad f_1' \to 1.0; \quad f_2' \to 1.0; \quad g \to 1.0 \quad (2b)$$

Following the conventional notation, $C$ is the viscosity-density product normalized by the edge of boundary-layer value, $\rho$ is the density with the subscript $e$ denoting edge condition, $f_1$ and $f_2$ are the stream function in the meridional and transverse directions, respectively, and $E_1$ is the ratio of the inviscid flow kinetic energy to the total flow energy. Primes note differentiation with respect to

$$\eta = \left(\frac{3\rho_e u_{1,e}}{2\mu_e x_1}\right)^{1/2} \int_0^{x_3} \rho/\rho_e \, d\bar{x}_3 \quad (3)$$

with $x_1$, $u_{1,e}$ being the meridional distance and inviscid velocity, respectively, and $\mu_e$ the viscosity at the boundary-layer edge. The problem is closed with the gas property relations which for the model gases were taken as:

$$p \alpha h^{-1}; \quad Pr = 0.6,\ 0.715,\ 0.9,\ 1.0; \quad \mu \alpha h^{\omega} \quad \omega = 0.5,\ 0.7,\ 1.0 \quad (4)$$

with $h$ being the static enthalpy. For real gases the correlations of Cohen[8] were used. Exact solutions were obtained using the method developed in Ref. 9 with the same computer code as was used in Ref. 10, where its accuracy and reliability were demonstrated. Computation of an average problem took approximately 3 sec of IBM 360/65 computer time.

The cross-flow is characterized by:

$$\bar{\alpha} = \frac{2}{3}\frac{\partial u_{2,e}/\partial\theta}{u_{1,e}\sin\delta_c} = \frac{1}{3}\left[\left(1 - \frac{2\partial^2 p/\partial\theta^2}{\rho_e u_{1,e}^2 \sin^2\delta_c}\right)^{1/2} - 1\right] \quad (5)$$

where $\theta$ is the meridional angle, $\delta_c$ the cone half angle and $p$ the pressure. For engineering estimates it is convenient to have a simple relation for $\bar{\alpha}$, even at the expense of some loss of accuracy. With the Newtonian flow relations developed by Leigh and Ross[11] transverse velocity gradients may be computed directly and

$$\bar{\alpha} = \frac{1}{3}\left[\left(1 + \frac{8(p_0 - p)\sin\alpha\cos\delta_c\sin(\alpha+\delta_c)}{\rho_e u_{1,e}^2 \sin^2\delta_c}\right)^{1/2} - 1\right] \quad (6)$$

with $p_0$ being the local stagnation pressure and $\bar{\alpha}$ the angle of attack. If we now make a further approximation by assuming that the Mach number at the most windward generator is equal to the local Mach number on a cone with the half angle of $(\alpha + \delta_c)$ then the conical flow charts and gas dynamics tables of Ref. 12 may be used to estimate $\bar{\alpha}$. For instance with $E_1 = 0.5$, $\alpha = 2\delta_c$ and $\alpha$ small, $\bar{\alpha}$ is readily estimated to be 2.5. From the graphs in Ref. 14 it is seen that a surface Mach number of about 2.2 corresponds to a 9° cone in an air stream at Mach 2.4 so that in this case we have $\delta_c = 3°$, $\alpha = 6°$. Equivalently $E_1 = 0.5$, $\bar{\alpha} = 2.5$ are found on an 8.5° cone at 18° angle of attack in an airstream at Mach 3.5. These approximate estimates, based on Newtonian flow, may not be accurate for low incidence angles of the leading cone generator, but are probably adequate for first estimates. Exact inviscid solutions are available for such cases.

The basic, zero injection, heat-transfer data are given in Table 1, in the form of heat transfer normalized by the zero angle-of-attack value, $q_s^*/q_{s,0}^*$. The asterisk superscript denotes no injection, subscript $s$ denotes surface condition, and the subscript 0 refers to the zero angle-of-attack situation. It should be noted that $q_s^*/q_{s,0}^*$ is virtually independent of $\omega$ but increases slowly with $E_1$ and $g_s$, the wall to total enthalpy ratio. Thus, for a given set of $E_1$ and $g_s$, the calculations performed for simplified gas models may be related to realistic gas properties results through the zero angle-of-attack correlation functions. The situation changes completely with mass transfer and as shown in Fig. 1, gas properties play a significant role in the effectiveness of injection.

In Table 2 the no mass transfer recovery factors $r^*$ are seen to increase slightly with $\bar{\alpha}$ and decrease with $E_1$. Recovery factors normalized by their no injection values $r/r^*$ are displayed in Table 3 to show their weak dependence on $\bar{\alpha}$ and $E_1$.

**Table 2 Recovery factor $r^*$, $\omega = 0.5$, $Pr = 0.715$**

| $E_1$ | $\bar{\alpha} = 0.0$ | 0.25 | 0.50 | 1.00 | 2.00 | 3.00 |
|---|---|---|---|---|---|---|
| 0.5 | 0.839 | 0.845 | 0.847 | 0.850 | 0.852 | 0.853 |
| 0.7 | 0.836 | 0.843 | 0.846 | 0.850 | 0.851 | 0.852 |
| 0.9 | 0.829 | 0.842 | 0.846 | 0.850 | 0.852 | 0.853 |

**Table 3 Reduction of $r$ by injection, $\omega = 0.5$, $Pr = 0.715$**

| | $r/r^*$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | $E_1 =$ 0.5 | 0.5 | 0.5 | 0.5 | 0.7 | 0.7 | 0.9 |
| $-f_s$ | $\bar{\alpha} =$ 0.5 | 1.0 | 2.0 | 3.0 | 3.0 | 1.0 | 1.0 |
| 0.0 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.2 | 0.978 | 0.982 | 0.987 | 0.989 | 0.989 | 0.981 | 0.981 |
| 0.5 | 0.943 | 0.955 | 0.966 | 0.971 | 0.971 | 0.953 | 0.952 |
| 1.0 | 0.888 | 0.910 | 0.931 | 0.941 | 0.941 | 0.908 | 0.908 |
| 2.0 | 0.857 | 0.861 | 0.877 | 0.891 | 0.891 | 0.861 | 0.865 |

In view of the weak dependence of $r^*$ on $\bar{\alpha}$, it is interesting to check the suggestion of Reshotko who, on the basis of his $C = 1.0$ results, suggested that $r = Pr^{1/2}$. The data in Table 4 support Reshotko's conjecture.

**Table 4 Influence of $Pr$ on $r$, $\omega = 0.5$, $\bar{\alpha} = 2.0$, $E_1 = 0.5$**

| | | | |
|---|---|---|---|
| $Pr$ | $= 0.6$ | 0.715 | 0.9 |
| $r$ | $= 0.783$ | 0.839 | 0.951 |
| $(Pr)^{1/2}$ | $= 0.775$ | 0.846 | 0.949 |

With the demonstrated weak influence of gas properties on $q_s^*/q_{s,0}^*$ similar results for the meridional and transverse shear stresses are expected and only typical data are displayed in Table 5.

**Table 5 Influence of gas properties on the shear stresses**

| $Pr$ / $\omega$ | 0.715 / 0.5 | 0.715 / 0.7 | 0.715 / 1.0 | 1.00 / 1.00 | 10,000 fps | 20,000 fps |
|---|---|---|---|---|---|---|
| $\bar{\alpha}$ | | a) $\tau_{1,s}^*/\tau_{1,s,0}^*$, $g_s = 0.1$, $E = 0.5$ | | | | |
| 0.0 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.5 | 1.30 | 1.30 | 1.29 | 1.30 | 1.30 | 1.31 |
| 1.0 | 1.54 | 1.53 | 1.53 | 1.54 | 1.54 | 1.55 |
| 2.0 | 1.92 | 1.92 | 1.91 | 1.94 | 1.94 | 1.95 |
| 3.0 | 2.25 | 2.24 | 2.23 | 2.26 | 2.26 | 2.28 |
| $\bar{\alpha}$ | | b) $\tau_{2,s}^*/\tau_{2,s,0}^*$, $g_s = 0.1$, $E = 0.5$ | | | | |
| 0.0 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.5 | 1.28 | 1.28 | 1.28 | 1.28 | 1.28 | 1.28 |
| 1.0 | 1.51 | 1.51 | 1.50 | 1.50 | 1.51 | 1.51 |
| 2.0 | 1.89 | 1.88 | 1.88 | 1.88 | 1.89 | 1.89 |
| 3.0 | 2.20 | 2.20 | 2.19 | 2.19 | 2.20 | 2.21 |

The most interesting observation here is that the normalized meridional shear stresses $\tau_{1,s}^*/\tau_{1,s}^*$ are almost identical with the corresponding normalized heat-transfer rates. Normalized transverse shear stresses follow this observation to within a few

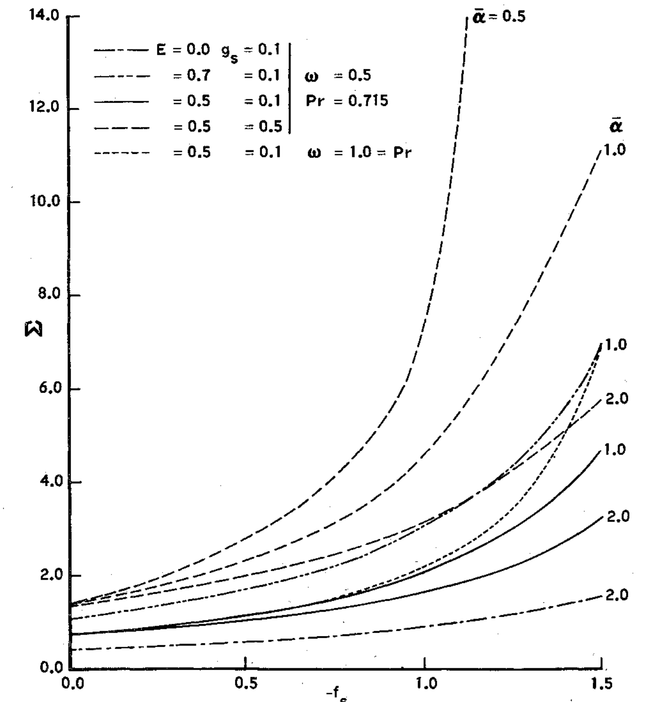


Fig. 2 **Influence of injection on the skewing of the flow within the boundary layer.**

percent. In Table 6, the influence of the flow energy parameter is displayed to complete the data.

**Table 6 Influence of $E_1$ on $\tau_{1,s}^*/\tau_{1,s,0}^*$, $\omega = 0.5$, $Pr = 0.715$, $g_s = 0.1$**

| $\bar{\alpha}$ | $E = 0.0$ | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|
| 0.0 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.5 | 1.26 | 1.30 | 1.34 | 1.49 |
| 1.0 | 1.48 | 1.54 | 1.00 | 1.77 |
| 2.0 | 1.84 | 1.92 | 2.02 | 2.28 |
| 3.0 | 2.03 | 2.25 | 2.36 | 2.68 |

We now have sufficient data to estimate accurately the effects of Mach number, wall to total enthalpy ratio and angle of attack on the meridional shear stress and heat transfer to the most windward meridians of cones at angles of attack.

Finally, with the view of evaluating the validity of the small cross-flow assumption, we turn to the purely kinematic aspects of the flow. If we define skewness of the flow Σ as the difference between the tangents of the flow angles across the boundary layer (i.e., $u_{2,s}/u_{1,s} - u_{2,e}/u_{1,e})/(u_{2,e}/u_{1,e})$, then,

$$\Sigma = f_{2,s}''/f_{1,s}'' - 1 \tag{7}$$

Selected data are shown in Fig. 2 to indicate the strong effect of injection on Σ. These results suggest that the small cross-flow assumption should be reexamined in situations involving injection.

This Note demonstrates that normalization of heat-transfer and shear stress data by the zero angle-of-attack values eliminates the influence of gas properties. Further, it is shown that the normalized heat-transfer and meridional shear stress values are almost identical so that the more readily available experimental heat-transfer data may be used to estimate shear stress. A wide range of conditions is considered to provide sufficient data for engineering estimates.

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# Nonstationary Response Exceedance Statistics of a Simple Mechanical System


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### Introduction

THIS Note concerns the response of a single degree-of-freedom system to a type of nonstationary random excitation. The parameter of particular interest is the expected frequency of exceedances of given levels by the displacement response. The motion of the system is governed by the familiar equation

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = y(t)/m \tag{1}$$

and the excitation or forcing function is of the form

$$y(t) = n(t)g(t) = n(t)(1 + \beta \sin \omega t)\left[U(t - t_0) - U(t - t_f)\right] \tag{2}$$

where $n(t)$ is stationary Gaussian noise with zero mean and $U(t)$ denotes the unit step function. It is assumed that the system is stable with less than critical damping ($0 < \zeta < 1$), and that the noise is exponentially correlated, i.e.,

$$E\{n(t_1)n(t_2)\} = R_{nn}(t_2 - t_1) = \sigma_n^2 \exp\left[-\alpha|t_2 - t_1|\right] \tag{3}$$

The present work has been motivated by a study of the response of aircraft to atmospheric turbulence. The aircraft is usually modeled as a linear system and turbulent forces as


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Index category: Aircraft Gust Loading and Wind Shear.



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